

Turbulent Instability of Anti-de Sitter Space

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joint work with Piotr Bizoń

Phys.Rev.Lett.107 031102 (2011) (arXiv:1104.3702)

now a part of joint project with Joanna Jałmużna, Patryk Mach and Maciej Maliborski

Warszawa, 2nd October 2012

Maximally symmetric solutions of vacuum Einstein's equations and their stability

$$R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0.$$

- $\Lambda = 0$: Minkowski (trivial, yet most important)
asymptotically stable (Christodoulou&Klainerman 1993),
- $\Lambda > 0$: de Sitter (important in cosmology - Nobel Prize 2011)
asymptotically stable (Friedrich 1986),
- $\Lambda < 0$: anti- de Sitter (most popular on arXiv due to AdS/CFT)
Stability of AdS seems unexplored

Anti-de Sitter spacetime in $d + 1$ dimensions

Geometrically, AdS_{d+1} is wrapped around hiperboloid

$$-(X^0)^2 + \sum_{k=1}^d (X^k)^2 - (X^{d+1})^2 = -\ell^2$$

embedded in flat, $SO(d, 2)$ invariaint space

$$ds^2 = -(dX^0)^2 + \sum_{k=1}^d (dX^k)^2 - (dX^{d+1})^2$$

Parametrization

$$X^0 = \sqrt{r^2 + \ell^2} \cos(\tau/\ell), \quad X^{d+1} = \sqrt{r^2 + \ell^2} \sin(\tau/\ell)$$

$$X^k = rn^k, \quad 0 \leq r < \infty, \quad \sum_{k=1}^d n^k = 1,$$

induces

$$ds^2 = - \left(1 + \frac{r^2}{\ell^2} \right) d\tau^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\Omega_{S^{d-1}}^2$$

Anti-de Sitter spacetime in $d + 1$ dimensions

AdS is the maximally symmetric solution of the vacuum Einstein equations $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = 0$ with negative cosmological constant $\Lambda < 0$:

$$ds^2 = - \left(1 + \frac{r^2}{\ell^2} \right) d\tau^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\Omega_{S^{d-1}}^2, \quad 0 \leq r, \quad -\infty < \tau < \infty$$

$$\Lambda = -d(d-1)/(2\ell^2)$$

Substituting $r = \ell \tan x$ and $\tau = \ell t$ we get

$$ds^2 = \frac{\ell^2}{(\cos x)^2} \left[-dt^2 + dx^2 + (\sin x)^2 d\Omega_{S^{d-1}}^2 \right],$$

$$-\infty < t < +\infty, \quad 0 \leq x < \pi/2.$$

Conformal infinity $x = \pi/2$ is the timelike hypersurface $\mathcal{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds_{\mathcal{I}}^2 = -dt^2 + d\Omega_{S^{d-1}}^2$

Is AdS stable?

- A solution (of a dynamical system) is said to be stable if small perturbations of it at $t = 0$ remain small for all later times
- The question of stability of AdS is open. Surprisingly, with almost fifteen years of activity on AdS/CFT, this question has been rarely asked (with a notable exception [M. Anderson 2005](#))
- In contrast, Minkowski ($\Lambda = 0$) and de Sitter spacetimes ($\Lambda > 0$) are known to be stable (actually asymptotically stable) – ([Christodoulou&Klainerman 1993](#) and [Friedrich 1986](#))
- The key difference between these solutions and AdS: the main mechanism of stability - dissipation of energy (dispersion in Minkowski, expansion in de Sitter) - is absent in AdS because AdS is effectively bounded (for no flux boundary conditions at \mathcal{I} it acts as a perfect cavity)
- Note that by positive energy theorems both Minkowski and AdS are the unique ground states among asymptotically flat/AdS spacetimes

Model

- To deal with the problem of the stability of AdS we start with spherical symmetry (effectually 1 + 1 dimensional problem)
- Spherically symmetric vacuum solutions are static (Birkhoff's theorem) \Rightarrow we need matter to generate dynamics
- Simple matter model: massless scalar field ϕ in $d+1$ dimensions

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial_\mu \phi \partial^\mu \phi \right), \quad \Lambda = -d(d-1)/(2\ell^2)$$
$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0$$

- In the corresponding asymptotically flat ($\Lambda = 0$) model Christodoulou proved the weak cosmic censorship (dispersion for small data and collapse to a black hole for large data) and Choptuik discovered critical phenomena at the threshold for black hole formation
- Remark: For even $d \geq 4$ there is a way to bypass Birkhoff's theorem (cohomogeneity-two Bianchi IX ansatz, Bizoń, Chmaj, Schmidt 2005)

- Convenient parametrization of asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{(\cos x)^2} \left[-Ae^{-2\delta} dt^2 + A^{-1} dx^2 + (\sin x)^2 d\Omega_{S^{d-1}}^2 \right],$$

where A and δ are functions of (t, x) .

- Auxiliary variables $\Phi = \phi'$ and $\Pi = A^{-1} e^{\delta} \dot{\phi}$ ($' = \partial_x, \dot{} = \partial_t$)
- Field equations (using units where $8\pi G = d - 1$)

$$A' = (1 - A) \frac{d - 2 + 2(\sin x)^2}{(\cos x)(\sin x)} - (\cos x)(\sin x) A (\Phi^2 + \Pi^2),$$

$$\delta' = -(\cos x)(\sin x) (\Phi^2 + \Pi^2),$$

$$\dot{\Phi} = (Ae^{-\delta}\Pi)', \quad \dot{\Pi} = \frac{1}{(\tan x)^{d-1}} \left[(\tan x)^{d-1} Ae^{-\delta}\Phi \right]'$$

- AdS space: $\phi \equiv 0, A \equiv 1, \delta \equiv 0$; now we want to perturb AdS solving the initial-boundary value problem for this system starting with some small, smooth initial data $(\phi, \dot{\phi})|_{t=0}$

Boundary conditions

- We assume that initial data $(\phi, \dot{\phi})|_{t=0}$ are smooth
- Smoothness at the center implies that near $x = 0$ (Λ irrelevant)
$$\phi(t, x) = f_0(t) + \mathcal{O}(x^2), \quad \delta(t, x) = \mathcal{O}(x^2), \quad A(t, x) = 1 + \mathcal{O}(x^2)$$
- Smoothness at spatial infinity and conservation of the total mass M imply that near $x = \pi/2$ (using $z = \pi/2 - x$)

$$\phi(t, x) = f_\infty(t) z^d + \mathcal{O}(z^{d+2}), \quad \delta(t, x) = \delta_\infty(t) + \mathcal{O}(z^{2d}),$$
$$A(t, x) = 1 - (M/\ell^{d-2})z^d + \mathcal{O}(z^{d+2})$$

Remark: There is **no freedom in prescribing boundary data**

- Local well-posedness (Friedrich 1995, Holzegel&Smulevici 2011)
- mass function and asymptotic mass:

$$m(t, x) = (1 - A(t, \rho)) (\ell \tan x)^{d-2} (1 + \tan^2 x)$$

$$M = \lim_{x \rightarrow \pi/2} m(t, x) = \ell^{d-2} \int_0^{\pi/2} (A\Phi^2 + A\Pi^2) (\tan x)^{d-1} dx$$

Reminder: asymptotically flat ($\Lambda = 0$) self-gravitating scalar field

- Christodoulou (1986-1993): dispersion for small data and collapse to a black hole for large data (proof of the weak cosmic censorship)
- Consider a family of initial data $\Phi(p)$ which interpolates between dispersion and collapse (Choptuik 1993)
- There exists a critical value of the parameter p^* such that
 - ▶ $p < p^* \Rightarrow$ dispersion
 - ▶ $p > p^* \Rightarrow$ black hole
- Universal behavior in the near-critical region $|p - p^*| \ll 1$
 - ▶ $m_{BH} \sim (p^* - p)^\gamma$ with universal exponent γ
 - ▶ discretely self-similar attractor with universal period Δ
- Critical solution ($p = p^*$) is a non-generic naked singularity

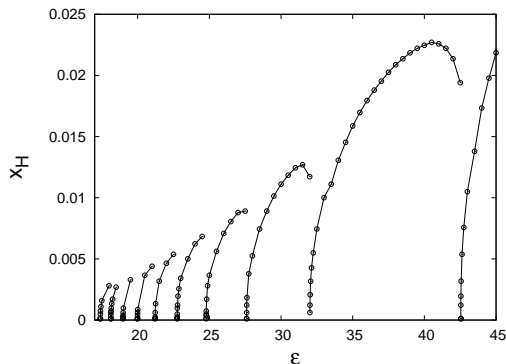
Movie

- from now $d = 3$
- qualitatively the same behaviour in any $d \geq 3$
- $d = 2$ case is very special (Pretorius&Choptuik 2000)

Critical behavior

Initial data: $\Phi(0, x) = 0$, $\Pi(0, x) = \varepsilon \left[\exp\left(-\frac{\tan x}{\sigma}\right)^2 \right]$

We fix $\sigma = 1/16$ and vary ε .



BH size vs. amplitude

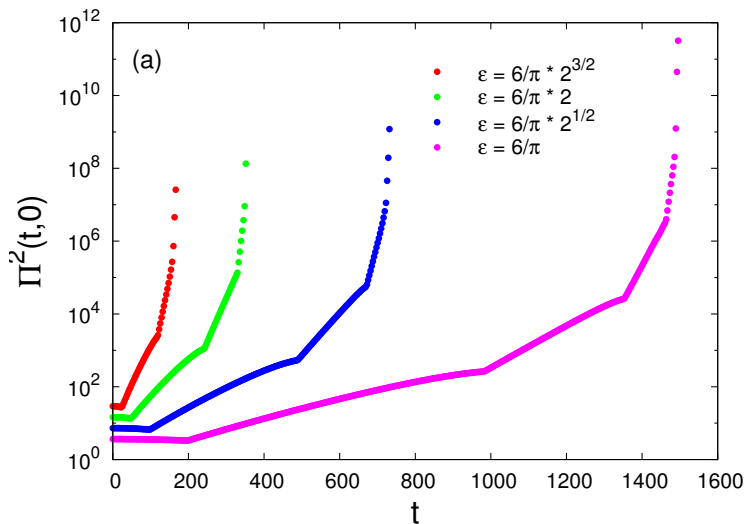
Remark: The generic endstate of evolution is the Schwarzschild-AdS BH of mass M (in accord with [Holzegel&Smulevici 2011](#))

There is a decreasing sequence of critical amplitudes ε_n for which the evolution, after making n reflections from the AdS boundary, locally asymptotes Choptuik's solution. In each small right neighborhood of ε_n

$$m_{BH}(\varepsilon) \sim (\varepsilon - \varepsilon_n)^\gamma$$

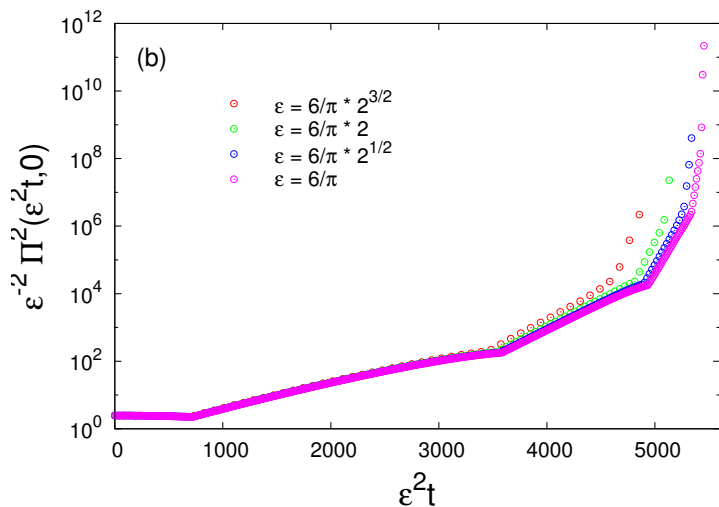
with $\gamma \simeq 0.37$. It seems that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

Key evidence for instability



$$\text{Ricci scalar } R = 2(\Phi^2 - \Pi^2) / \ell^2 - 12 / \ell^2$$

Key evidence for instability



Onset of instability at time $t = \mathcal{O}(\epsilon^{-2})$

Weakly nonlinear perturbations

- We seek an approximate solution starting from small initial data $(\phi, \dot{\phi})|_{t=0} = (\varepsilon f(x), \varepsilon g(x))$
- Perturbation series

$$\phi = \varepsilon \phi_1 + \varepsilon^3 \phi_3 + \dots$$

$$\delta = \varepsilon^2 \delta_2 + \varepsilon^4 \delta_4 + \dots$$

$$1 - A = \varepsilon^2 A_2 + \varepsilon^4 A_4 + \dots$$

where $(\phi_1, \dot{\phi}_1)|_{t=0} = (f(x), g(x))$ and $(\phi_j, \dot{\phi}_j)|_{t=0} = (0, 0)$ for $j > 1$.

- Inserting this expansion into the field equations and collecting terms of the same order in ε , we obtain a hierarchy of linear equations which can be solved order-by-order.

First order

- Linearized equation (Ishibashi&Wald 2004)

$$\ddot{\phi}_1 + L\phi_1 = 0, \quad L = -\frac{1}{\tan^2 x} \partial_x (\tan^2 x \partial_x)$$

The operator L is essentially self-adjoint on $L^2([0, \pi/2), \tan^2 x dx)$.

- Eigenvalues and eigenvectors of L are ($j = 0, 1, \dots$)

$$\omega_j^2 = (3 + 2j)^2, \quad e_j(x) = N_j (\cos x)^3 F \left(\begin{matrix} -j, 3+j \\ 3/2 \end{matrix} \middle| (\sin x)^2 \right)$$

\Rightarrow AdS is linearly stable

- Linearized solution

$$\phi_1(t, x) = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j) e_j(x)$$

where amplitudes a_j and phases β_j are determined by the initial data.

Second order (back-reaction on the metric)

$$A_2' + \frac{1 + 2 \sin^2 x}{\sin x \cos x} A_2 = \sin x \cos x \left(\dot{\phi}_1^2 + \phi_1'^2 \right)$$
$$\delta_2' = -\sin x \cos x \left(\dot{\phi}_1^2 + \phi_1'^2 \right)$$

so

$$A_2(t, x) = \frac{\cos^3 x}{\sin x} \int_0^x \left(\dot{\phi}_1(t, y)^2 + \phi_1'(t, y)^2 \right) \tan^2 y \, dy$$

$$\delta_2(t, x) = - \int_0^x \left(\dot{\phi}_1(t, y)^2 + \phi_1'(t, y)^2 \right) \sin y \cos y \, dy$$

quadratic in ϕ_1 !

(1)

It follows that

$$M = \frac{\varepsilon^2}{2} \int_0^{\pi/2} \left(\dot{\phi}_1(t, y)^2 + \phi_1'(t, y)^2 \right) \tan^2 y \, dy + \mathcal{O}(\varepsilon^4)$$

Third order

- $\ddot{\phi}_3 + L\phi_3 = S(\phi_1, A_2, \delta_2), \quad (\star)$
where $S := 2(A_2 + \delta_2)\ddot{\phi}_1 + (\dot{A}_2 + \dot{\delta}_2)\dot{\phi}_1 + (A'_2 + \delta'_2)\phi'_1$.

- Let $\phi_3(t, x) = \sum_j c_j(t) e_j(x)$. Projecting Eq.(\star) on the basis $\{e_j\}$ we obtain an infinite set of decoupled forced harmonic oscillators for the generalized Fourier coefficients $c_j(t) := (e_j, \phi_3)$

$$\ddot{c}_j + \omega_j^2 c_j = S_j := (e_j, S) \quad \text{and} \quad (c_j, \dot{c}_j)|_{t=0} = 0$$

If S_j has a part oscillating with a resonant frequency ω_j it will give rise to a secular term in c_j , that is $c_j \sim t \sin \omega_j t$ or $c_j \sim t \cos \omega_j t$.

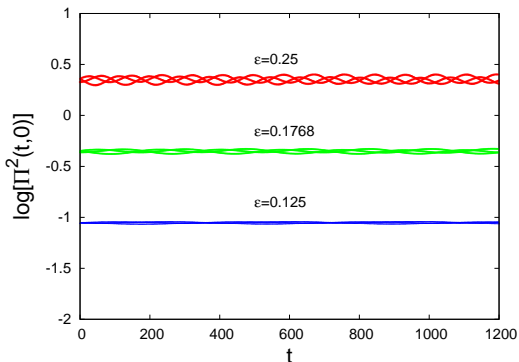
- **S cubic in ϕ_1** \Rightarrow contains all frequencies $|\pm \omega_1 \pm \omega_2 \pm \omega_3|$, where $\omega_k \in \Omega_1$ and $\phi_1(t, x) = \sum_k [\omega_k \in \Omega_1] a_k \cos(\omega_k t + \beta_k) e_k(x)$, ω_k - odd integers \Rightarrow all frequencies in S potentially resonant!
Not all resonances survive the projection (e_j, S) . Some of those, which do survive can be compensated with frequency shifts in ϕ_1 and are harmless for stability, but the others put stability in question!

Example 1: single-mode data $\phi(0, x) = \varepsilon e_0(x)$

- First order $\phi_1(t, x) = \cos(\omega_0 t) e_0(x)$, $\omega_0 = 3$ ($\omega_j = 3 + 2j$)
- Third order $\phi_3(t, x) = \sum_{j=0}^{\infty} c_j(t) e_j(x)$, $(c_j, \dot{c}_j)|_{t=0} = 0$ and

$$\ddot{c}_j + \omega_j^2 c_j = b_{j,0} \cos(\omega_0 t) + b_{j,3} \cos(\omega_3 t).$$

But $b_{3,3} = 0$ (!) and only $j = 0$ is resonant.



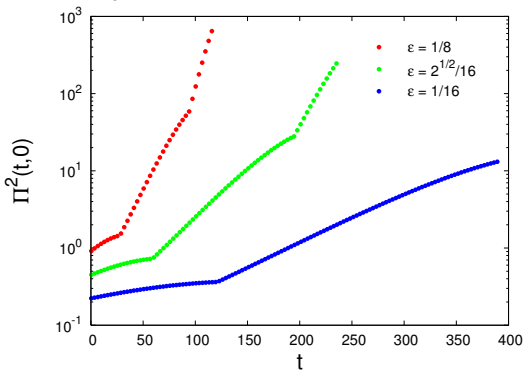
The $j = 0$ resonance can be easily removed by the two-scale method (slow-time phase modulation) which gives $\phi_1 = \cos((\omega_0 + \frac{153}{4\pi} \varepsilon^2) t) e_0(x)$. This suggests that there are non-generic initial data which may stay close to AdS solution

Example 2: two-mode data $\phi(0, x) = \varepsilon (e_0(x) + e_1(x))$

- First order $\phi_1(t, x) = \cos(\omega_0 t)e_0(x) + \cos(\omega_1 t)e_1(x)$, $\omega_0 = 3$, $\omega_1 = 5$
- Third order $\phi_3(t, x) = \sum_{j=0}^{\infty} c_j(t)e_j(x)$, $(c_j, \dot{c}_j)|_{t=0} = 0$ and

$$\ddot{c}_j + \omega_j^2 c_j = \sum_k [\omega_k \in \Omega_3] b_{j,k} \cos(\omega_k t), \quad \text{where } \Omega_3 = \{|\pm\omega_{0,1} \pm \omega_{0,1} \pm \omega_{0,1}\}$$

Here $\Omega_3 = \{1, 3, 5, 7, 9, 11, 13, 15\}$, but the resonance ($b_{j,j} \neq 0$) only if $\omega_j \in \{3, 5, 7\}$.



$\omega_0 \rightarrow \omega_0 + (87/\pi)\varepsilon^2$,
 $\omega_1 \rightarrow \omega_1 + (413/\pi)\varepsilon^2$ shifts
 remove the resonances
 $\omega_j = 3, 5$, but the resonance
 $\omega_j = 7$ cannot be removed.
 Thus we get the secular term
 $c_2(t) \sim t \sin(7t)$. We expect
 this term to be a progenitor of
 the onset of exponential
 instability.

Conjectures

Our numerical and formal perturbative computations lead us to:

Conjecture 1

Anti-de Sitter space is unstable against the formation of a black hole under arbitrarily small generic perturbations

Proof (and a precise formulation) is left as a challenge. Note that we do **not** claim that all perturbed solutions end up as black holes.

Conjecture 2

There are non-generic initial data which may stay close to AdS solution; Einstein-scalar-AdS equations may admit time-quasiperiodic solutions

Proof: KAM theory for PDEs?

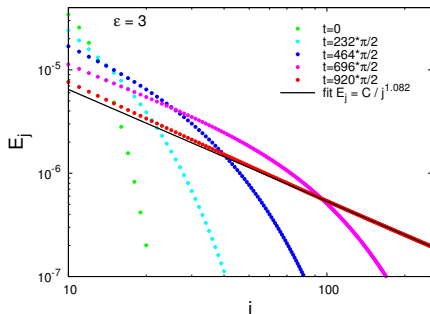
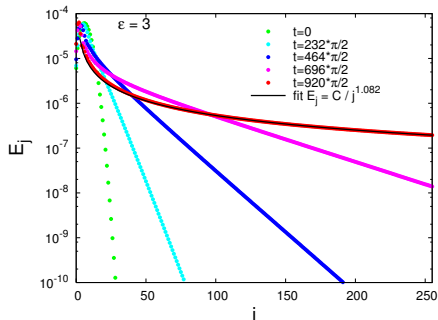
Analogous conjecture for vacuum Einstein's equations by [Dias, Horowitz and Santos \(2011\)](#) (existence of geons)

Turbulence: transfer of energy from low to high frequencies

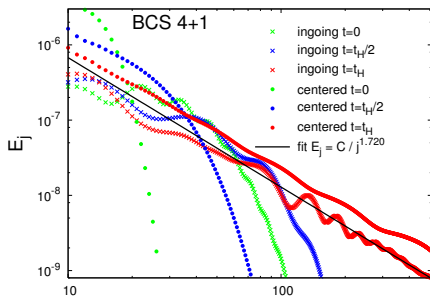
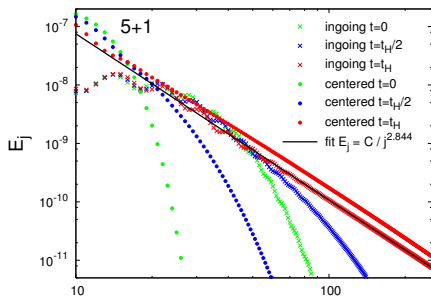
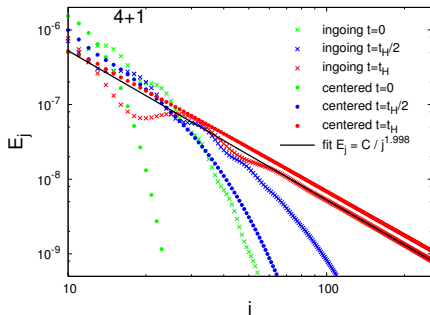
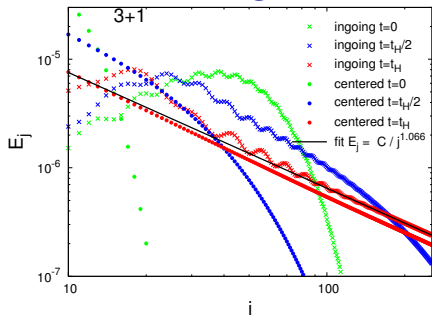
Let $\Pi_j := (\sqrt{A}\Pi, e_j)$ and $\Phi_j := (\sqrt{A}\Phi, e_j')$. Then

$$M = \ell \int_0^{\pi/2} (A\Phi^2 + A\Pi^2) (\tan x)^2 dx = \sum_{j=0}^{\infty} E_j(t),$$

where $E_j := \Pi_j^2 + \omega_j^{-2}\Phi_j^2$ can be interpreted as the j -mode energy.



Power-law scaling



Final remarks

- Weakly turbulent behavior seems to be common for (non-integrable) nonlinear wave equations on bounded domains (e.g. NLS on torus, Colliander, Keel, Staffilani, Takaoka, Tao 2008, Carles, Faou 2010) and our work shows that Einstein's equations are not an exception.
- For Einstein's equations the transfer of energy to high frequencies cannot proceed forever because concentration of energy on smaller and smaller scales inevitably leads to the formation of a black hole.
- The role of negative cosmological constant is purely kinematical, that is the only role of Λ is to confine the evolution in an effectively bounded domain. Similar turbulent dynamics has been observed for small perturbations of Minkowski in a box (Maliborski 2012)
- Generalizations: different matter models, "deformed" boundary conditions, relaxing symmetry, multi-scale analysis, (in)stability of AdS black holes (Holzegel & Smulevici 2011, Horowitz 2012), instability of AdS_{2+1} (work in progress by Jałmużna)