Oscillatory singularities

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Background

- 4D GR.
- Generic spacelike oscillatory singularities for vacuum and perfect fluids with soft equations of state.

- Lifshitz and Khalatnikov 1963: Singularities are not generic.
- Penrose 1965: Singularities are generic.
- Bianchi type IX oscillatory singularities : Misner 1969, Belinskii, Khalatnikov and Lifshitz (BKL) 1970.
- BKL 1982: Generic spacelike oscillatory singularities.

Oscillations of what?

- ”BKL’s” asymptotic locality conjecture = asymptotically each spatial point evolves toward the singularity individually and independently of its neighbors as a spatially homogeneous model.
- Asymptotic vacuum dominance = the spacetime geometry is asymptotically described by Einstein’s vacuum equations toward a spacelike singularity, even though e.g. matter quantities like the energy density might blow up.
  - Why? A gravitational competition between matter and pure gravity in a generic context. Examples suggest that causal properties are essential; matter sources with characteristics < the speed of light
    are vacuum dominated?

- BKL: Sequences of vacuum Bianchi type I (Kasner) and Bianchi type II solutions describe the evolution along individual timelines.

Kasner, i.e. vacuum Bianchi type I, models.

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2$$

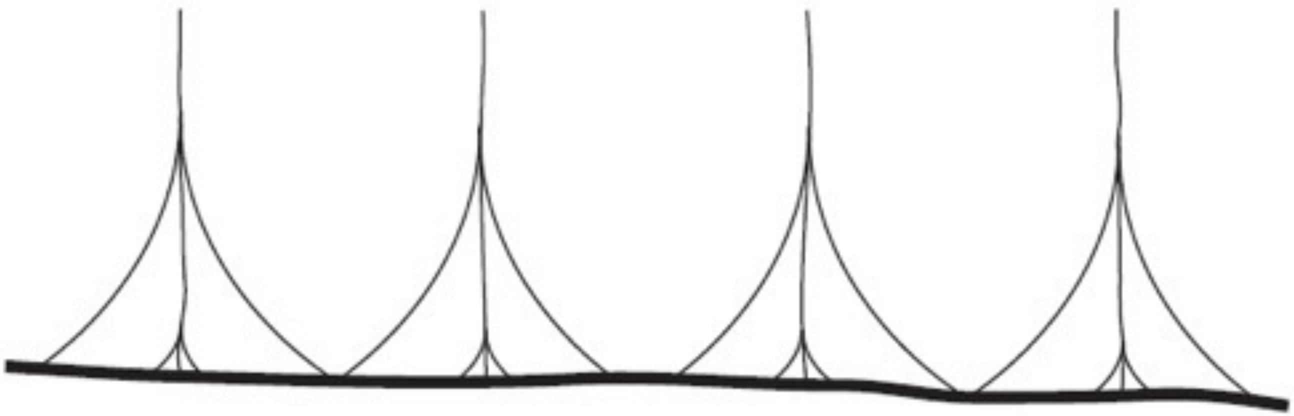
$$p_1 + p_2 + p_3 = 1 \,, \quad p_1^2 + p_2^2 + p_3^2 = 1$$

$$p_1 p_2 p_3 = \frac{-u^2(1+u)^2}{(1+u+u^2)^3} \qquad u \in [1, \infty]$$

New developments

- Discovery of formulations of Einstein’s field equations that allows for steps towards rigorous formulations of BKL.
- Discovery of some underlying reasons for <u>why</u> BKL happens, and how BKL is related to hierarchical structures in GR.
- Discovery of theorems in spatially homogeneous (SH) cosmology, in particular for Bianchi type IX.
- Discovery that BKL is not always true: ”BKL locality breaking” at certain (surfaces of) timelines via temporally transient recurring oscillatory spikes, but both behaviours are characterized by oscillatory Kasner states.
- Discovery that the building blocks for BKL oscillations and non-BKL spike oscillations are closely related, e.g.

Asymptotic causal structure



- Asymptotic silence: A necessary, but not sufficient, condition for BKL locality?

Asymptotic state space regularization: A new context

- Proximate goal: Construction of an asymptotic approximation of a generic solution in a small neighborhood of a spacelike singularity.
- Assume that there exists a foliation in that neighborhood with leaves that asymptotically coincides with the singularity.
- Use a characteristic dimensional variable scale of the problem, the expansion (Hubble variable) of the normal congruence of the foliation to conformally blov $g_{ab} = H^{-2}\bar{g}_{ab}\,, \qquad H = \frac{1}{3}\theta = -\frac{1}{3}\,\mathrm{tr}\,k$

- Use of a conformally Hubble-normalized frame. Variables: 1/H and the dimensionless Hubble-normalized orthonormal spatial frame and connection variables.
- → Asymptotically regularized first order PDE version of Einstein's field equations for BKL and recurring spike behaviour.
- = Hubble-normalized dynamical systems formulation when spatial homogeneity is imposed + decoupled equations for 1/H and the Hubble-normalized spatial frame variables.
- Setting the spatial frame variables to zero gives the SH equations on an invariant boundary subset – ”the local boundary” → SH/local boundary correspondense.
- Constants of integration are replaced with spatial functions on the local boundary = a precise asymptotically regularized state space setting for BKL!

Hierarchical state space structures

- Local and partially local boundaries, especially the partially local boundary associated with two commuting spacelike Killing vectors = the partially local G_2 boundary → symmetry-boundary correspondense
- The Lie contraction hierarchy on the local boundary = SH models due to the SH/local boundary correspondense.
- The scale-autmorphism hierarchy on the local boundary:
  - Kinematics → coupled system with true degrees of freedom.
  - Dynamics → monotone functions and conserved quantities. The dynamical scale-autmorphism hierarchy limits asymptotic dynamics toward the singularity hierarchically to boundaries of boundaries, eventually determined by conserved quantities, as is the case for Kasner (vacuum Bianchi type I) and Bianchi type II (hierarchical

- Much of asymptotic dynamics is hence a consequence of the physical first principles of scale-invariance and general covariance.

- How much of the properties of extreme gravity is determined by physical first principles?

Hierarchical solution generation

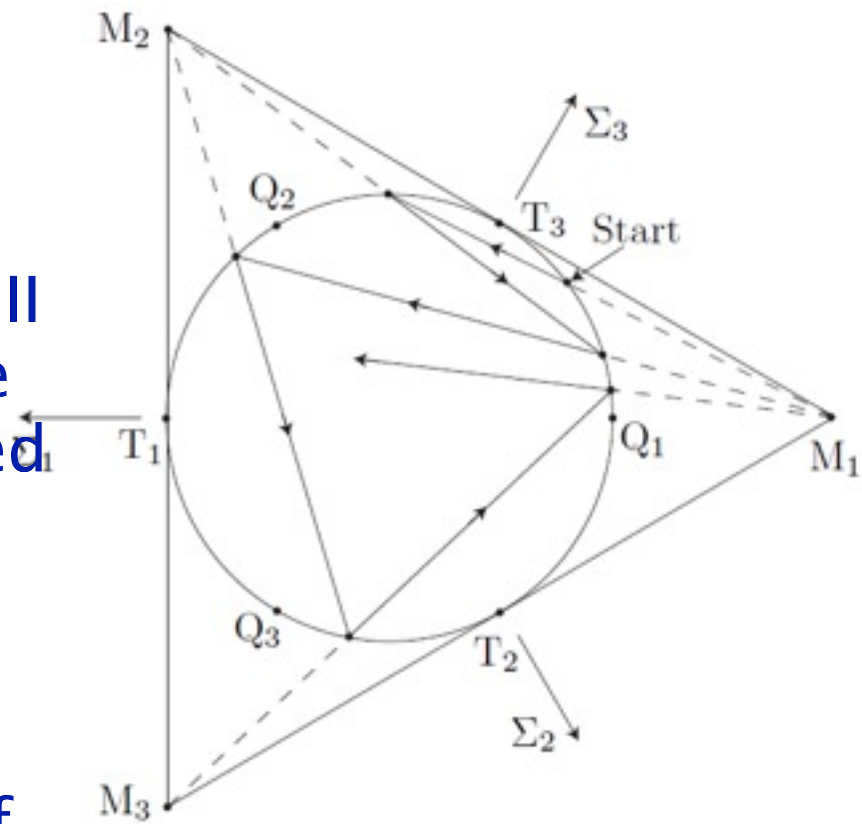
- Rendall and Weaver (2001): Applied a solution generating technique to Fuchsian asymptotic expansions in T^3 Gowdy (asymptotically non-oscillatory) models → asymptotic expansions for “true” and “false” spikes.
- Lim (2008) used the solution generating algorithm to produce a hierarchic sequence of one-parameter families of exact solutions:
  Kasner → ”rotating ” Kasner → Bianchi type II → rotating Bianchi type II (= false spike solution) → spike solutions → … (all are solutions corresponding to the orthogonally transitive case of models with two commuting Killing vectors).

Concatenation, chains and

- On the local boundary:

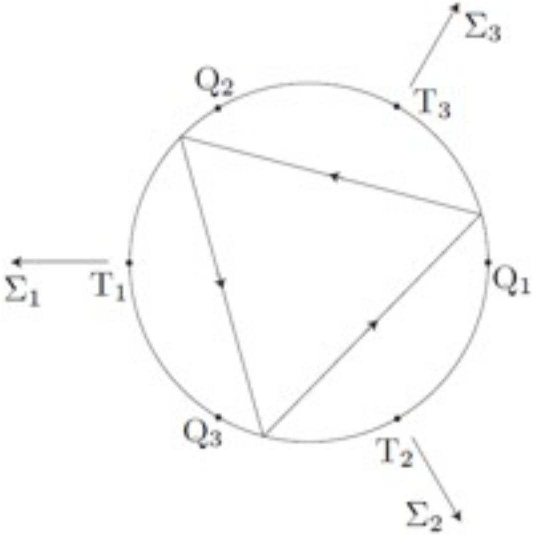
$$\Sigma_1 := \frac{k^1{}_1 - \frac{1}{3}\operatorname{tr}k}{\frac{1}{3}\operatorname{tr}k}, \quad \Sigma_2 := \frac{k^2{}_2 - \frac{1}{3}\operatorname{tr}k}{\frac{1}{3}\operatorname{tr}k}, \quad \Sigma_3 := \frac{k^3{}_3 - \frac{1}{3}\operatorname{tr}k}{\frac{1}{3}\operatorname{tr}k}; \quad \Sigma_1 + \Sigma_2 + \Sigma_3 = 0$$

- The Kasner circle of fixed points: Self-similarity, Rosquist, Jantzen (1985).
- Concatenation of Bianchi type II orbits via Kasner points on the Kasner circle (= the generalized Kasner "solution" on the local boundary) into heteroclinic chains = asymptotic temporal matching into chains.
  Gives rigor to BKL's concept of piecewise solutions.



- The recurring Bianchi type II orbits represent gravito–elektromagnetic oscillations that connect different purely gravito–electric Kasner states, thus leading to Kasner oscillations.
- When the Kasner circle of fixed points on the local boundary is completely destabilized (by sufficiently many destabilizingly ”trigger” gravito–electromagnetic degrees of freedom), generic chains are infinitely long toward the past, leading to infinite Kasner oscillations.
- Proof by Ringström (2000, 2001) that Bianchi types VIII and IX have oscillatory singularities, and that type IX has an attractor residing on

$$\mathcal{A}_{\rm IX} \in \overline{\mathbf{B}}_{\rm II} := \mathrm{K}^{\bigcirc} \cup \mathcal{B}_{N_1} \cup \mathcal{B}_{N_2} \cup \mathcal{B}_{N_3}$$



(also Heinzle and Uggla

Chain discretization I: BKL and Misner: Change of Kasner states by means of a sequence of vacuum Bianchi type II solutions → the "Kasner map" (toward the past singularity)

$$u^{\mathrm{f}} = \begin{cases} u^{\mathrm{i}} - 1 & \text{if} \quad u^{\mathrm{i}} \in [2, \infty)\,, \\ (u^{\mathrm{i}} - 1)^{-1} & \text{if} \quad u^{\mathrm{i}} \in [1, 2]\,, \end{cases}$$

$$\ldots \to 1.14 \to \underbrace{7.29 \to 6.29 \to 5.29 \to 4.29 \to 3.29 \to 2.29 \to 1.29}_{\text{era}} \to \underbrace{3.45 \to 2.45 \to 1.45}_{\text{era}} \to \underbrace{2.24 \to 1.24}_{\text{era}} \to \ldots$$

u_s = maximum value of u in era s

$$\mathrm{u}_s = k_s + x_s \qquad k_s = [\mathrm{u}_s]\,, \qquad x_s = \{\mathrm{u}_s\}$$

(BKL) Era map: $\mathrm{u}_s \mapsto \mathrm{u}_{s+1}$

$$\mathrm{u}_{s+1} = \frac{1}{x_s} = \frac{1}{\{\mathrm{u}_s\}}$$

$$\mathrm{u}_0 = k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cdots}} = [k_0; k_1, k_2, k_3, \ldots]$$

Maps and chaotic properties... Asymptotic dynamics?

Asymptotic type VIII and IX dynamics,
u classification (Heinzle and Uggla 2009):

1. $u_0 = [k_0; k_1, k_2, \ldots, k_n]$ ($u_0 \in Q$) Ringström (2000) proved that such finite sequences are not asymptotically realized!

2. $u_0 = [k_0; k_1, \ldots]$ bounded with or without periodicity. Without periodicity Béguin (2010) proved that a family of solutions of codimension one converges to each associated chain. Liebscher, Härterich, Webster, Georgi (2011) proved that a family of solutions of codimension one converges to the cycles associated with $u_0 = [(1)]$, and argued that this holds for the

Inhomogeneous cosmology

- No general rigorous inhomogneous dynamical oscillatory singularity results.
- Analytical, heuristic analytic analysis, numerical results.
- However, note that all rigorous spatially homogeneous dynamical results translate into results on the local boundary = BKL dynamics, thus obtaining possible relevance in a generic inhomogeneous context (if a solution approach the local boundary in a spatially asymptotically differentiable way).
- BKL breaking: Recurring spikes.

- On the partially local G_2 boundary:
  - True spikes are generated when a ”gravito-electromagnetic trigger degree of freedom” goes through zero at a spatial coordinate on the partially local G_2 boundary.
  - True spikes belong to an invariant subset on the G_2 boundary – ”the spike subset”, which is naturally described in terms of spatial (Hubble-normalized) Iwasawa frames.
  - Concatenation of spike solutions. In contrast to BKL concatenation, this involves families of timelines, where (combinations of) spike solutions are joined at points on the Kasner circle on the local boundary, thus forming an analogue to heteroclinic chains. (The spike solutions play an analogous role to the generalized Bianchi type II solutions on the local boundary in the BKL picture.)

- As in the BKL case, if there are sufficiently many degrees of freedom, which here also includes Hubble-normalized spatial frame derivatives, that destabilize the Kasner circle on the local boundary, generic recurring spike chains are infinitely long toward the past, leading to infinite Kasner oscillations.
- The spikes are gravito-elektromagnetic in nature, centered at gravito-elektric spike surfaces.
- The spatial size of the spikes are shrinking asymptotically toward zero toward the spike surface.

- On the partially local G_1 boundary:
  - Spike surface intersections that form temporary lines?
- In the general G_0 state space:
  - Spike line intersections that form temporary spike

Chain discretization II: Kasner maps

- The Mixmaster state space map.
- The Iwasawa state space map.
- The BKL Kasner map (in terms of the gauge invariant Kasner parameter u).
- The BKL Kasner era map.

- The spike Kasner map = the BKL Kasner map applied twice.
- The spike Kasner era map.
- The Iwasawa state space spike map.

- Beyond the local boundary:
  Heuristic perturbations of the local boundary (which illustrates the possibility of ”small/ close to generalized Kasner initial data” in the Hubble-normalized state space picture) suggests:
  - Consistency of the BKL (and billiard) picture(s).
  - Bianchi type IX is too symmetric to show all relevant ”BKL” features.
  - There seems to be both ”local dynamical” (as in the type IX attractor approach) and ”historical statistical” (as in Bianchi type VIII and $VI_{-1/9}$?) suppression of degrees of freedom to the infinite heteroclinic chain boundaries on the local boundary.

- Numerics suggest consistency of the BKL picture, but that it also needs to be complemented with recurring spikes, i.e., different timelines of generic solutions seem to be attracted both to the heteroclinic chains on the local boundary and to the invariant ”spike subset” on the partially local G_2 boundary.
- Explicit spike solutions are needed to provide ”spike zooming” in order to obtain numerically valid results: With hindsight, earlier numerical experiments produce and annihilate recurring spikes artificially.
- Special models without sufficiently many Kasner circle destabilizing trigger degrees of freedom ”interrupt and mutilate” recurring spikes to create permanent spikes.
- The non-uniform features of permanent spikes are very different from the non-uniform features that are expected from infinitely recurring spike formation.

The past attractor

- The past attractor for generic spacelike singularities consists of two parts only?
  1. The union of the Kasner and Bianchi type II subsets on the local boundary.
  2. The union of the Kasner subset on the local boundary and the spike subset on the partially local G_2 boundary.

Open issues concerning recurring spikes

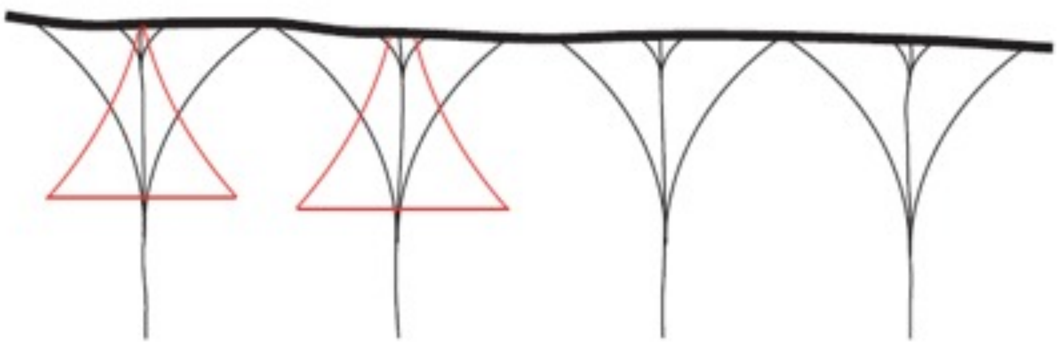
- Moving spikes and spike freezing?
- Spike formation and annihilation and the asymptotic number of spikes?
- Spike crossing?
- The physical reasons for spike formation?

The physical context of generic spacelike singularities

- Not cosmology but black hole formation? (Talk with bad title?)
  - Open set of initial data forming black holes. Is the same really true for cosmology?
  - A connection between almost axisymmetry in black hole formation and the axisymmetric "dominant" Taub points?
  - The Taub points belong to the Taub subset that is associated with phenomena such as caustics, horizons, and weak null singularities.
  - Generic spacelike singularities are "almost weak null singularities"?
  - Generic spacelike singularities are part of black hole singularities that also involve weak null singularities?

The dangers and possibilities of special

- Are special models like the Gowdy models more misleading than helpful?
- The importance of topological “timing”.



- Does Bianchi type VIII and $VI_{-1/9}$ describe more generic features for generic spacelike singularities than Bianchi type IX?
- The issue of extracting generically relevant information from non-generic situations.
- Heuristics and numerics as guidelines for finding and choosing physically relevant toy models and provide a physically and mathematically progressive research

Billiards and beyond the GR context

- The ”billiard projection” of Misner, Chitre, Damour, Henneaux and Nicolai as a dual projection to the Hubble-normalized shear projection.
- Hamiltonian methods and billiards and scale-invariant dynamical systems methods in GR and beyond give quite different pictures, use the associated synergy! Historical example, the discovery of monotone functions.
- Hierarchical state space structures connected with BKL and spike formation, solution generating algorithms, Hamiltonians and billiards as tools for finding asymptotic hidden symmetries and asymptotically conserved quantities.
- Spatially homogeneous models and asymptotic silence?
- Asymptotic quantization of the generic strong gravity regime?